

# Entanglement annihilating and entanglement breaking channels

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**Abstract.** We introduce and investigate a family of entanglement-annihilating channels. These channels are capable to destroy any quantum entanglement within the system they act on. We show that they are not necessarily entanglement-breaking. In order to achieve this result we analyze the subset of locally entanglement-annihilating channels. In this case, same local noise applied on each subsystem individually is less entanglement-annihilating (with respect to multi-partite entanglement) as the number of subsystems is increasing. Therefore, bipartite case provides restrictions on the set of local entanglement-annihilating channels for multipartite case. The introduced concepts are illustrated on the family of single-qubit depolarizing channels.

Submitted to: *J. Phys. A: Math. Gen.*

## 1. Introduction

The phenomenon of quantum entanglement [1] was recognized as an important resource in many applications of quantum information theory [2]. The parallel power of quantum computers as well as the security of quantum cryptosystems relies on the peculiar properties of entangled states of composite systems. For example, Shor's algorithm [3], or quantum teleportation [4], could not be invented and successful without the puzzling properties of quantum entanglement.

By definition, *entanglement* is a property assigned to multipartite quantum states. Following Werner [5], we say that a state  $\omega$  of some bipartite system is separable, if it can be expressed as a convex combination of factorized states, i.e. written in the form  $\omega = \sum_j p_j \xi_j \otimes \zeta_j$ . If it cannot, we say it is *entangled*. Entangled states exhibit their nonlocal origin in the following sense. Spatially separated experimentalists cannot create entanglement without some exchange of quantum systems, i.e. only by local actions and classical communication.

The concepts of entanglement and separability can be directly generalized to the multipartite case. Moreover, in this case a more subtle “entanglement-induced”

separation of the state space is possible. For example, the so-called GHZ state  $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$  is an example of an entangled three-partite state, but no pair of subsystems is mutually entangled, because each pair is described by the classically correlated state  $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ . Qualitatively different family [6] of three-partite states is represented by a state  $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ . In this case, each pair of subsystems is entangled. This illustrates that the entanglement theory of multipartite systems is more complex and represents an interesting field of research.

The tasks related to detection and characterization of entanglement represent prominent problems of quantum entanglement theory [7, 8]. In this paper, we will pay attention to entanglement dynamics induced by the evolution of individual quantum subsystems. Since perfectly isolated quantum systems are very difficult to achieve in real experiments, unavoidable noise affects the states and potentially causes changes in the shared entanglement. It is of importance to understand the robustness of entanglement with respect to these (local) processes.

For example, in order to perform the GHZ experiment [9] the three-partite entangled state  $|\text{GHZ}\rangle$  must be distributed to the laboratories of Gina, Helen and Zoe. However, it is very likely that the transmissions over long distances are not perfect and Gina, Helen and Zoe will actually receive and work with systems described by a modified quantum state  $\omega'_{\text{GHZ}}$ . Also, the storage of quantum systems and performing the experiments themselves represent additional sources of noise. Depending on the particular type of overall noise the modified state may or may not be still used to perform the intended experiment and observe a certain phenomenon.

The better we understand the dynamical robustness of entanglement, the better we can perform multipartite experiments. Entanglement is the quantum resource, but it seems to be very fragile, which reduces our abilities to conduct scalable (in any sense) experiments. For this purposes its distribution, storage and careful local manipulation is important. Loosely speaking, our goal is to separate the "bad" noise from noise that is relatively "nice". In other words, which types of environmental influences must Gina, Helen and Zoe try to avoid, and which ones are acceptable? Intriguing questions are related to experiments with an increasing number of parties. Namely, are there local channels destroying any entanglement completely? How does it depend on the number of parties? For a given local channel is there always a number of parties, for which its action on each individual subsystem completely destroys any shared entanglement? Under the presence of arbitrary local noise is there any limit on the number of particles that can be entangled? Such questions are partially addressed in this paper.

The interplay between the entanglement we created and its dynamical stability under particular sources of noise is currently a vivid field of research. Zanardi et al. in [10, 11] asked the question which unitary transformations are better in creating entanglement. Linden et al. have shown in [12] that capacities of unitary channels to create and destroy entanglement are not the same. In particular, there are unitary channels that can create (on average) more entanglement than they can destroy. Zyczkowski et al. [13] analyzed the dynamics of entanglement for various different

models of nonunitary evolutions. They showed the basic qualitative features how entanglement evolves in time. Since then the phenomenon known as entanglement sudden death attracted relatively many researchers (see [14] and references therein) who analyzed many dynamical models and made many observations concerning the entanglement dynamics. In [15, 16, 17] it was shown that local channels do not preserve entanglement-induced ordering. After its action originally more entangled states can become less entangled than states coming from some originally less entangled states. Recently, an evolution equation (in fact inequality) for the entanglement affected by local independent noise has been formulated [18, 19, 20].

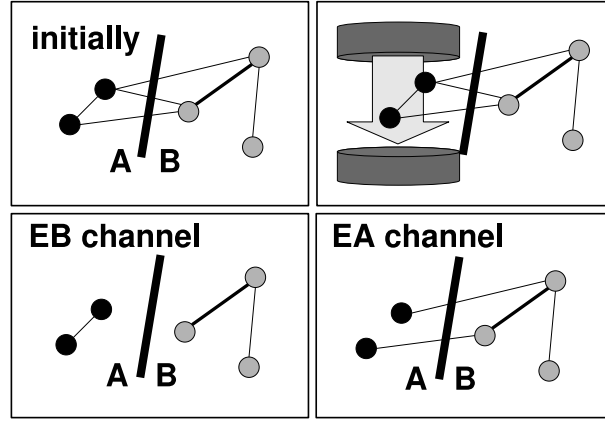
In this paper we focus on channels that completely destroy any entanglement. Clearly, unitary channels do not possess such property. For them entanglement creation goes always in hand with entanglement annihilation. However, the situation becomes more interesting when general nonunitary evolutions are considered. In Section II we present our definitions and list the basic properties of the so-called *entanglement-breaking* and *entanglement-annihilating* channels. From the perspective of these concepts, in Section III we investigate the channels acting locally on a multipartite composite system. In Section IV we study a family of depolarizing channels in more detail. Finally, we summarize our observations in Section V.

## 2. Preliminaries

A composite quantum system  $Q$  consisting of  $n$  quantum systems is associated with a Hilbert space  $\mathcal{H}_Q \equiv \mathcal{H}^{(n)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ . Its states are represented by so-called density operators, i.e. positive operators with a unit trace. Let us denote by  $\mathcal{S}(\mathcal{H}) = \{\varrho : \varrho \geq 0, \text{tr}[\varrho] = 1\}$  the set of all states of a system associated with the Hilbert space  $\mathcal{H}$ . We divide the system  $Q$  into two subsystems  $A$  and  $B$  consisting of  $k$  and  $n - k$  particles with Hilbert spaces  $\mathcal{H}_A = \mathcal{H}^{(k)}$ ,  $\mathcal{H}_B = \mathcal{H}^{(n \setminus k)}$ , respectively. When denoting the total Hilbert space as  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  we mean that the whole system is understood as a bipartite system consisting of subsystems  $A$  and  $B$ .

Any Hilbert space  $\mathcal{H}$  with a defined tensor structure we can divide into two subsets  $\mathcal{S}_{\text{ent}}(\mathcal{H}), \mathcal{S}_{\text{sep}}(\mathcal{H})$  of entangled and separable states. In particular,  $\mathcal{S}_{\text{sep}}(\mathcal{H}_Q)$  is the set of all separable states with respect to the division of  $\mathcal{H}_Q$  into  $n$  particles ( $n$ -partite separability), i.e. it consists of states of the form  $\varrho = \sum_j p_j \varrho_1^{(j)} \otimes \cdots \otimes \varrho_n^{(j)}$ . Similarly, the set  $\mathcal{S}_{\text{sep}}(\mathcal{H}_A)$  contains separable states of  $k$  particles forming the subsystem  $A$ . However, we will use  $\mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$  to denote the set of separable states with respect to division of  $\mathcal{H}_{AB}$  into subsystems  $A$  and  $B$  (bipartite separability), i.e. this set consists of states that can be expressed as  $\varrho = \sum_j p_j \varrho_A^{(j)} \otimes \varrho_B^{(j)}$ . In such case the internal structure of the composite subsystems  $A$  and  $B$  is irrelevant and  $\mathcal{S}_{\text{sep}}(\mathcal{H}_Q) \subsetneq \mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$  (meaning  $\mathcal{S}_{\text{sep}}(\mathcal{H}_Q)$  is a strict subset of  $\mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$ ). The analogous notation will be used for subsets of entangled states  $\mathcal{S}_{\text{ent}}(\mathcal{H}_Q)$ ,  $\mathcal{S}_{\text{ent}}(\mathcal{H}_A)$ , and  $\mathcal{S}_{\text{ent}}(\mathcal{H}_{AB})$ .

The evolution of quantum systems is described by means of quantum channels, i.e. completely positive trace-preserving linear maps  $\mathcal{E}$  defined on the set of all linear



**Figure 1.** The action of entanglement-breaking and entanglement-annihilating channel is illustrated. Lines between the systems exhibits the existence of entanglement.

operators  $\mathcal{L}(\mathcal{H})$  on the considered Hilbert space  $\mathcal{H}$ . A linear mapping  $\mathcal{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  defines a quantum channel if  $\text{tr}[\mathcal{E}[X]] = \text{tr}[X]$  for all  $X \in \mathcal{L}(\mathcal{H})$  and  $(\mathcal{E} \otimes \mathcal{I})[X]$  remains a positive operator for all positive operators  $X \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_{\text{anc}})$ , where  $\mathcal{H}_{\text{anc}}$  is the Hilbert space associated with an ancillary system of arbitrary size. We use  $\mathcal{I}$  to denote the identity (trivial) channel on the ancillary system. In what follows we will denote the ancillary system by  $B$ , thus, in our further consideration the subsystem  $B$  can be of arbitrary size and structure.

**Definition 1.** We say the channel  $\mathcal{E}_A$  acting on the subsystem  $\mathcal{H}_A$  is

- *entanglement-annihilating* (EA) if

$$\mathcal{E}_A[\mathcal{S}(\mathcal{H}_A)] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_A).$$

- *entanglement-breaking* (EB) if

$$\mathcal{E}_A \otimes \mathcal{I}_B[\mathcal{S}(\mathcal{H}_{AB})] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$$

for arbitrary ancillary system  $B$ .

Thus, the entanglement-annihilating (EA) channels are defined as the ones that completely destroy/annihilate any entanglement within the subset  $A$  of the composite system (see Fig. 2). On contrary, the entanglement-breaking (EB) channels are those that completely disentangle the subsystem they are acting on from the rest of the system. Note that by definition the EA channels (acting on subsystem  $A$ ) do not necessarily disentangle the subsystems  $A$  and  $B$ . Similarly, the EB channels do not necessarily destroy entanglement within the subsystem  $A$ . The two concepts are thus (by definition) different and our aim is to investigate their mutual relationship.

Let us denote by  $\mathcal{T}(\mathcal{H})$  the set of all linear maps  $\mathcal{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ . Let  $\mathcal{T}_{\text{chan}}(\mathcal{H})$  be the set of all channels on system associated with a Hilbert space  $\mathcal{H}$ , i.e.  $\mathcal{T}_{\text{chan}}(\mathcal{H}_A)$  are all channels defined on subsystem  $A$ . Let  $\mathcal{T}_{\text{EA}}(\mathcal{H}_A)$  and  $\mathcal{T}_{\text{EB}}(\mathcal{H}_A)$  denote the subsets of EA and EB channels, respectively.

### 2.1. Basic properties

In quantum theory, measurements are associated with so-called positive operators valued measures (POVMs), i.e. collections of positive operators  $F_1, \dots, F_n$  such that  $\sum_j F_j = I$ . As it was shown in Ref. [21] any entanglement-breaking channel can be understood as a *measure and prepare* procedure. That is, each EB channel can be expressed in the form

$$\mathcal{E}_A[\cdot] = \sum_j \text{tr}[\cdot F_j] \varrho_j, \quad (1)$$

for some POVM  $\{F_j\}$  and some fixed states  $\varrho_1, \dots, \varrho_n$ .

To verify whether a given channel is entanglement-breaking or not, is equivalent to detecting whether a specific bipartite quantum state is separable, or entangled. Let us denote by  $P_+ = |\psi_+\rangle\langle\psi_+|$  a projector onto the maximally entangled vector state  $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_j |\varphi_j \otimes \varphi_j\rangle$  of Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ , where  $\mathcal{H}_B \equiv \mathcal{H}_A$ , and vectors  $\{|\varphi_j\rangle\}_j$  form an orthonormal basis of the Hilbert space  $\mathcal{H}_A$  of dimension  $d$ . A mapping  $\mathcal{J} : \mathbb{T}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$  defined via the identity [22, 23, 24]

$$\mathcal{J}(\mathcal{E}_A) = (\mathcal{E}_A \otimes \mathcal{I}_B)[P_+] \equiv \Omega_{\mathcal{E}}, \quad (2)$$

determines a unique operator  $\Omega_{\mathcal{E}}$  for each linear mapping  $\mathcal{E} \in \mathbb{T}(\mathcal{H}_A)$ . It is known as the Choi-Jamiolkowski isomorphism. The Choi-Jamiolkowski operator  $\Omega_{\mathcal{E}}$  provides an alternative representation of a quantum channel (acting on the system associated with the Hilbert space  $\mathcal{H}_A$ ) as a specific linear operator on the Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . The complete positivity of  $\mathcal{E}_A$  is translated to the positivity of  $\Omega_{\mathcal{E}}$  and the trace-preserving condition is equivalent with  $\text{tr}_A \Omega_{\mathcal{E}} = \frac{1}{d} I$ , thus  $\Omega_{\mathcal{E}}$  is a valid density operator on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Clearly, if  $\mathcal{E}_A$  is entanglement-breaking, then  $\Omega_{\mathcal{E}}$  is a separable state on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Surprisingly, the inverse implication is also true [21], i.e. the separability of  $\Omega_{\mathcal{E}}$  is necessary and sufficient for  $\mathcal{E}_A$  being entanglement-breaking. This significantly simplifies the analysis of channels with respect to entanglement-breaking, because it is sufficient to test the action of the channel only on a single state – the maximally entangled state and test whether  $\Omega_{\mathcal{E}}$  is separable, or not.

While entanglement-breaking channels had been already investigated, the concept of entanglement-annihilating channels is new. An interesting question is how to test whether a given channel is entanglement-annihilating, or not. If  $\mathcal{E}[P_\psi]$  with  $P_\psi = |\psi\rangle\langle\psi|$  is separable for all pure states  $|\psi\rangle \in \mathcal{H}$ , then for any state  $\omega \in \mathcal{S}(\mathcal{H})$  the state  $\mathcal{E}[\omega]$  is separable as well. This follows from the fact that the set of separable states is convex and any state  $\omega$  can be decomposed into a convex combination of pure states, i.e.  $\omega = \sum_j p_j P_{\psi_j}$ . That is, whether, or not the channel is entanglement-annihilating, it is sufficient to test its action only on pure states. Unfortunately, this is still not an easy task. We left open whether there exists a simpler way of testing for the EA property.

Let us continue with simple observations on elementary properties of the sets of entanglement-annihilating and entanglement-breaking channels.

**Lemma 1.**  $\mathbb{T}_{\text{EA}}(\mathcal{H}_A), \mathbb{T}_{\text{EB}}(\mathcal{H}_A)$  are convex.

*Proof.* By definition, if  $\mathcal{E}_1, \mathcal{E}_2 \in \mathsf{T}_{\text{EA}}$ , then  $\mathcal{E}_j[\mathcal{S}(\mathcal{H}_A)] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_A)$  for  $j = 1, 2$ . Due to convexity of  $\mathcal{E}_j(\mathcal{S}(\mathcal{H}_A))$  and  $\mathcal{S}_{\text{sep}}(\mathcal{H}_A)$  it follows that also  $(\lambda\mathcal{E}_1 + (1-\lambda)\mathcal{E}_2)[\mathcal{S}(\mathcal{H}_A)] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_A)$ . Similarly for the case of EB channels.  $\square$

**Lemma 2.** *If  $\mathcal{E} \in \mathsf{T}_{\text{EA}}(\mathcal{H}_A)$  and  $\mathcal{F} \in \mathsf{T}(\mathcal{H}_A)$ , then  $\mathcal{E} \cdot \mathcal{F} \in \mathsf{T}_{\text{EA}}(\mathcal{H}_A)$ .*

*Proof.* Defining property of  $\mathcal{E}$  implies that  $\mathcal{E}[\mathcal{F}[\mathcal{S}(\mathcal{H}_A)]] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_A)$ , hence  $\mathcal{E} \cdot \mathcal{F}$  is an entanglement-annihilating channel.  $\square$

**Lemma 3.** *If  $\mathcal{E} \in \mathsf{T}_{\text{EB}}(\mathcal{H}_A)$  and  $\mathcal{F} \in \mathsf{T}(\mathcal{H}_A)$ , then  $\mathcal{E} \cdot \mathcal{F}, \mathcal{F} \cdot \mathcal{E} \in \mathsf{T}_{\text{EB}}(\mathcal{H}_A)$ .*

*Proof.* Since  $\mathcal{E} \in \mathsf{T}_{\text{EB}}(\mathcal{H}_A)$  it follows that  $(\mathcal{E} \otimes \mathcal{I})[(\mathcal{F} \otimes \mathcal{I})[\mathcal{S}(\mathcal{H}_{AB})]] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$ . For any  $\mathcal{F} \in \mathsf{T}(\mathcal{H}_A)$  the channel  $\mathcal{F} \otimes \mathcal{I}$  cannot create entanglement (out of separable state) between subsystems  $A$  and  $B$ . Therefore,  $(\mathcal{F} \otimes \mathcal{I})[(\mathcal{E} \otimes \mathcal{I})[\mathcal{S}(\mathcal{H}_{AB})]] \subset \mathcal{S}_{\text{sep}}(\mathcal{H}_{AB})$ , hence the both channels  $\mathcal{E} \cdot \mathcal{F}, \mathcal{F} \cdot \mathcal{E}$  are entanglement breaking providing that one of them is.  $\square$

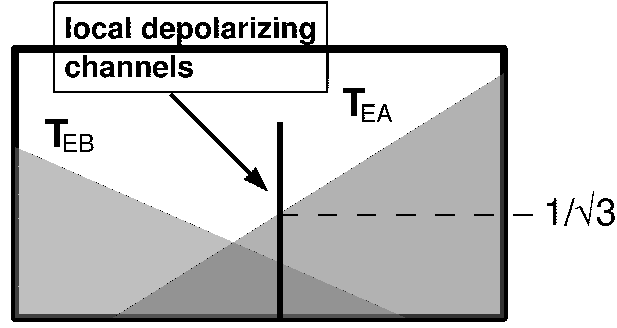
In what follows we will investigate the set relation between  $\mathsf{T}_{\text{EA}} \equiv \mathsf{T}_{\text{EA}}(\mathcal{H}_A)$  and  $\mathsf{T}_{\text{EB}} \equiv \mathsf{T}_{\text{EB}}(\mathcal{H}_A)$  (see Fig. 2), both defined on the same Hilbert space  $\mathcal{H}_A$ . As a consequence of the above lemmas we get that a composition  $\mathcal{E} \cdot \mathcal{F}$  of the entanglement-breaking channel  $\mathcal{F}$  and of the entanglement-annihilating channel  $\mathcal{E}$  belongs to the intersection  $\mathsf{T}_{\text{EA}} \cap \mathsf{T}_{\text{EB}}$ . That is, there are channels which are simultaneously EB and EA. On the other hand, although the channel  $\mathcal{F} \cdot \mathcal{E}$  is necessarily entanglement-breaking, it does not have to be entanglement-annihilating. For example, a single-point contraction  $\mathcal{F}$  of the whole state space into a single entangled state  $\omega \in \mathcal{S}_{\text{ent}}(\mathcal{H}_A)$  is entanglement-breaking, because it can be expressed in the form  $\mathcal{F}[\cdot] = \sum_j \text{tr}[\cdot F_j] \omega = \omega$ . However, the channel  $\mathcal{F} \cdot \mathcal{E}$  is not entanglement-annihilating for arbitrary  $\mathcal{E} \in \mathsf{T}_{\text{EA}}$ , because  $(\mathcal{F} \cdot \mathcal{E})[\mathcal{S}(\mathcal{H}_A)] = \omega \in \mathcal{S}_{\text{ent}}(\mathcal{H}_A)$ . This means there are entanglement-breaking channels which are not entanglement-annihilating, i.e.  $\mathsf{T}_{\text{EB}} \not\subset \mathsf{T}_{\text{EA}}$ . Later on we shall get back also to the inverse question whether  $\mathsf{T}_{\text{EA}} \subset \mathsf{T}_{\text{EB}}$ , or not.

**Lemma 4.** *Let  $\mathcal{E}[\cdot] = \sum_j \text{tr}[\cdot F_j] \varrho_j$ , i.e.  $\mathcal{E} \in \mathsf{T}_{\text{EB}}$ . Then the following statements hold:*

- (i) *If  $\varrho_j \in \mathcal{S}_{\text{sep}}(\mathcal{H}_A)$  for all  $j$ , then  $\mathcal{E} \in \mathsf{T}_{\text{EA}}$ .*
- (ii) *If there exists  $|\varphi\rangle \in \mathcal{H}_A$  such that  $F_j|\varphi\rangle = |\varphi\rangle$  for some  $j$ , then  $\mathcal{E} \in \mathsf{T}_{\text{EA}}$  only if  $\varrho_j$  is separable.*

*Proof.* The first part (i) is obvious, because convex sum of separable states is necessarily a separable state. The second part (ii) follows from the formula  $\mathcal{E}[|\varphi\rangle\langle\varphi|] = \varrho_j$ , which implies that being entanglement-annihilating requires that  $\varrho_j$  is separable.  $\square$

The second half of this lemma can be used to show that its first half can not be *if and only if* statement. Consider an entangled state  $\omega$ . Let us define  $\kappa$  as the largest value of  $x \in [0, 1]$  for which the state  $x\omega + (1-x)\frac{1}{d}I$  is separable. This value is strictly larger than 0. Let  $F$  be a positive operator with all the eigenvalues smaller than  $\kappa$ . Then  $\mathcal{E}[\cdot] = \text{tr}[\cdot F]\omega + \text{tr}[\cdot(I-F)]\frac{1}{d}I$  defines an entanglement-annihilating channel, because



**Figure 2.** This figure schematically illustrates the subsets of entanglement-annihilating and entanglement-breaking channels. The family of two-qubit local depolarizing channels  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  is depicted, too.

$\mathcal{E}[\varrho] = x\omega + (1-x)\frac{1}{d}I$  with  $x = \text{tr}[\varrho F] < \kappa$ . Thus,  $\mathcal{E} \in \mathcal{T}_{\text{EA}} \cap \mathcal{T}_{\text{EB}}$  does not imply that all  $\varrho_j$  are necessarily separable.

### 3. Local channels

We distinguish two basic types of channels acting on a composite system of  $k$  particles: global and local. We say a channel  $\mathcal{F}$  is local if it has a tensor product form  $\mathcal{F} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_k$ , where  $\mathcal{E}_j$  are channels acting on individual particles. If a channel does not have the factorized form we say it is global. In what follows we will investigate entanglement dynamics under local channels. Moreover, we will assume that Hilbert spaces of all particles are isomorphic and each particle undergoes the same evolution, i.e.  $\mathcal{E}_j = \mathcal{E}$  for all  $j$ . Although this is not the most general case, under certain circumstances it is of physical relevance.

We say a single-particle channel  $\mathcal{E}$  is a  $k$ -locally entanglement annihilating channel ( $k$ -LEA), if  $\mathcal{E}^{\otimes k} \in \mathcal{T}_{\text{EA}}(\mathcal{H}^{\otimes k})$ . Similarly,  $\mathcal{E}$  is a  $k$ -locally entanglement breaking channel ( $k$ -LEB) if  $\mathcal{E}^{\otimes k} \in \mathcal{T}_{\text{EB}}(\mathcal{H}^{\otimes k})$ . By  $\mathcal{T}_{k\text{-LEA}}$ ,  $\mathcal{T}_{k\text{-LEB}}$  we shall denote the subsets of  $k$ -LEA and  $k$ -LEB channels, respectively. Since elements of these sets are uniquely associated with single particle channels  $\mathcal{E} \in \mathcal{T}_{\text{chan}}(\mathcal{H})$ , we can understand these sets as subsets of  $\mathcal{T}_{\text{chan}}(\mathcal{H})$ , i.e.  $\mathcal{T}_{k\text{-LEA}}, \mathcal{T}_{k\text{-LEB}} \subset \mathcal{T}_{\text{chan}}(\mathcal{H})$ . Moreover, let us denote by  $\mathcal{T}_{\text{EB}}^1 \subset \mathcal{T}_{\text{chan}}(\mathcal{H})$  the subset of entanglement breaking channels acting on the single system  $\mathcal{H}$ , i.e.  $\mathcal{E} \in \mathcal{T}_{\text{EB}}^1$  means that  $(\mathcal{E} \otimes \mathcal{I}_{\text{anc}})[\omega]$  is separable for all  $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_{\text{anc}})$ . Our goal is to analyze the relation between the subsets  $\mathcal{T}_{k\text{-LEA}}$ ,  $\mathcal{T}_{k\text{-LEB}}$  and  $\mathcal{T}_{\text{EB}}^1$ . The ultimate question is which single particle channels  $\mathcal{E}$  if applied to a suitable number of particles, necessarily destroy any entanglement within the  $k$ -partite system.

By definition single-particle entanglement-breaking channels disentangle each particle from the rest of the system. Therefore, they are simultaneously locally entanglement-annihilating and locally entanglement-breaking channels for any value of  $k$ , hence

$$\mathcal{T}_{\text{EB}}^1 \subset \mathcal{T}_{k\text{-LEB}}; \quad \mathcal{T}_{\text{EB}}^1 \subset \mathcal{T}_{k\text{-LEA}}. \quad (3)$$

Further, consider a channel  $\mathcal{E} \in \mathsf{T}_{k\text{-LEA}}$ . It means  $\mathcal{E}^{\otimes k}$  transforms any state of  $k$  particles into some separable state of  $k$  particles, i.e.  $\mathcal{E}^{\otimes k}[\omega] = \sum_j p_j \xi_j^{(1)} \otimes \cdots \otimes \xi_j^{(k)}$ . Setting  $\omega = \varrho_0 \otimes \omega'$  we get that  $\mathcal{E}^{\otimes(k-1)}[\omega']$  is separable for any state  $\omega' \in \mathcal{S}(\mathcal{H}^{\otimes(k-1)})$ . That is,  $\mathcal{E}$  is also  $(k-1)$ -LEA channel. Consequently, we can write the relation

$$\mathsf{T}_{k\text{-LEA}} \subset \mathsf{T}_{l\text{-LEA}} \quad \text{for } k > l, \quad (4)$$

which implies

$$\mathsf{T}_{\text{EB}}^1 \subset \mathsf{T}_{\infty\text{-LEA}} \subset \cdots \subset \mathsf{T}_{3\text{-LEA}} \subset \mathsf{T}_{2\text{-LEA}}. \quad (5)$$

On the other hand for a channel  $\mathcal{E}^{\otimes k}$  the corresponding Choi operator takes the form  $\Omega_{\mathcal{E}}^{\otimes k}$ , where  $\Omega_{\mathcal{E}} = (\mathcal{E} \otimes \mathcal{I})[P_+]$ . If  $\mathcal{E}^{\otimes k}$  is entanglement breaking, i.e.  $\mathcal{E} \in \mathsf{T}_{k\text{-LEB}}$ , then  $\Omega_{\mathcal{E}}^{\otimes k}$  is separable with respect to a bipartite splitting into  $k$  principal systems and  $k$  ancillary systems. However, this implies that  $\Omega_{\mathcal{E}}$  itself is separable, hence, the single particle channel  $\mathcal{E}$  is entanglement breaking, i.e.  $\mathcal{E} \in \mathsf{T}_{\text{EB}}^1$ . As a result we get the following set identities

$$\mathsf{T}_{\text{EB}}^1 = \mathsf{T}_{2\text{-LEB}} = \mathsf{T}_{3\text{-LEB}} = \cdots = \mathsf{T}_{\infty\text{-LEB}}. \quad (6)$$

#### 4. A case study: depolarizing channels

In this section we will address the question whether the entanglement-breaking channels are not the only locally entanglement-annihilating channels. We will give explicit example of qubit channels that are not entanglement-breaking, but completely destroy entanglement if applied on individual particles.

Consider a one-parametric family of *depolarizing channels*

$$\mathcal{E}_{\lambda}[X] = \lambda X + (1 - \lambda) \text{tr}[X] \frac{1}{d} I, \quad (7)$$

where  $\lambda \in [0, 1]$ . Applying this channel to the maximally entangled state  $P_+$  we get the so-called Werner states [5]

$$\Omega_{\lambda} = \lambda P_+ + (1 - \lambda) \frac{1}{d} I \otimes \frac{1}{d} I. \quad (8)$$

For qubit ( $d = 2$ ) the states  $\Omega_{\lambda}$  are separable for  $\lambda \leq 1/3$ . Therefore, if  $\lambda \leq 1/3$  the qubit depolarizing channel  $\mathcal{E}_{\lambda}$  is entanglement-breaking.

##### 4.1. 2-LEA channels

The local channel  $\mathcal{E}_{\lambda} \otimes \mathcal{E}_{\lambda}$  acts as follows

$$\begin{aligned} \omega'_{12} &= (\mathcal{E}_{\lambda} \otimes \mathcal{E}_{\lambda})[\omega_{12}] = \lambda^2 \omega_{12} + (1 - \lambda)^2 \frac{1}{d} I \otimes \frac{1}{d} I \\ &\quad + \lambda(1 - \lambda) \left( \omega_1 \otimes \frac{1}{d} I + \frac{1}{d} I \otimes \omega_2 \right), \end{aligned} \quad (9)$$

where  $\omega_1 = \text{tr}_2[\omega_{12}]$  and  $\omega_2 = \text{tr}_1[\omega_{12}]$ . Unlike in the analysis of entanglement-breaking channels we need to verify the separability for all input states  $\omega_{12}$  in order to conclude that the channel  $\mathcal{E}_{\lambda}$  is 2-LEA. Fortunately, it is sufficient to analyze the separability



for pure states  $\omega_{12} = |\psi\rangle\langle\psi|$  only, because the set of separable states is convex and channels preserve the convexity. Let us note that  $\mathcal{E}_\lambda = \lambda\mathcal{I} + (1-\lambda)\mathcal{C}_0$ , where  $\mathcal{C}_0$  denotes the contraction of the whole state space into the complete mixture state  $\frac{1}{d}I$ . Both channels,  $\mathcal{I}, \mathcal{C}_0$  commutes with unitary transformations, i.e.  $\mathcal{I}[UXU^\dagger] = U\mathcal{I}[X]U^\dagger$  and  $\mathcal{C}_0[UXU^\dagger] = U\mathcal{C}_0[X]U^\dagger$ . Consequently,  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  commutes with all unitary channels  $U \otimes V$ . Using such local unitary channels any vector  $|\psi\rangle$  can be written in its Schmidt form

$$|\psi\rangle = \sum_j \sqrt{q_j} |\varphi_j\rangle \otimes |\varphi'_j\rangle, \quad (10)$$

where  $\{|\varphi_j\rangle\}, \{|\varphi'_j\rangle\}$  are suitable orthonormal bases of the first and the second particle, respectively. Because of the unitary invariance of  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  it is sufficient to consider only vectors expressed in a single fixed Schmidt basis. The reduced states take the diagonal form

$$\omega_1 = \sum_j q_j |\varphi_j\rangle\langle\varphi_j|, \quad \omega_2 = \sum_j q_j |\varphi'_j\rangle\langle\varphi'_j|. \quad (11)$$

Further, let us analyze the case of qubit, i.e.  $d = 2$  and  $|\psi\rangle = \sqrt{q_0}|\varphi_0\rangle \otimes |\varphi'_0\rangle + \sqrt{q_1}|\varphi_1\rangle \otimes |\varphi'_1\rangle$ , thus,

$$\omega'_{12} = \frac{1}{4} \begin{pmatrix} \lambda_-^2 + 4\lambda q_0 & 0 & 0 & 4\lambda^2 \sqrt{q_0 q_1} \\ 0 & \lambda_+ \lambda_- & 0 & 0 \\ 0 & 0 & \lambda_+ \lambda_- & 0 \\ 4\lambda^2 \sqrt{q_0 q_1} & 0 & 0 & \lambda_-^2 + 4\lambda q_1 \end{pmatrix},$$

where we set  $\lambda_\pm = 1 \pm \lambda$ . The eigenvalues of the partially transposed operator  $\omega'_{12}{}^\Gamma$  reads

$$\mu_1 = \frac{1}{4}(1 - \lambda)^2 + \lambda q_0; \quad (12)$$

$$\mu_2 = \frac{1}{4}(1 - \lambda)^2 + \lambda q_1; \quad (13)$$

$$\mu_\pm = \frac{1}{4}(1 - \lambda^2 \pm 4\lambda^2 \sqrt{q_0 q_1}); \quad (14)$$

where  $\lambda \in [0, 1]$ ,  $q_0 \in [0, 1]$  and  $q_1 = 1 - q_0$ . According to Peres-Horodecki criterion [25, 26] a two-qubit state  $\omega$  is separable if and only if  $\omega^\Gamma$  is positive. For larger systems this separability criterion is not sufficient to unambiguously distinguish between entangled and separable states. From nonpositivity of  $\omega^\Gamma$  we can conclude that the state is entangled, but the inverse implication does not hold.

All the eigenvalues of  $\omega'_{12}{}^\Gamma$  except  $\mu_-$  are always positive, hence it is sufficient to analyze only this one. In particular, if for a fixed value of  $\lambda$  the eigenvalue  $\mu_-$  is positive for all values  $q_0, q_1$ , then the corresponding depolarizing channel  $\mathcal{E}_\lambda$  is a 2-locally entanglement-annihilating channel. Thus, we want to minimize  $\mu_-$  over the interval  $q_0 \in [0, 1]$  for each depolarizing channel  $\mathcal{E}_\lambda$ . Fortunately, for each  $\lambda$  the minimum is achieved for the same value  $q_0 = q_1 = 1/2$ . Consequently, the partially transposed operator  $\omega'_{12}{}^\Gamma$  is positive if and only if  $1 - 3\lambda^2 \geq 0$ , i.e.

$$\lambda \leq \frac{1}{\sqrt{3}} \approx 0.577. \quad (15)$$

In summary, the qubit depolarizing channel  $\mathcal{E}_\lambda$  is 2-locally entanglement-annihilating if and only if  $\lambda \in [0, 1/\sqrt{3}]$ , whereas it is entanglement-breaking for  $\lambda \in [0, 1/3]$ , hence  $\mathsf{T}_{\text{EB}}^1 \neq \mathsf{T}_{2\text{-LEA}}$ .

#### 4.2. 3-LEA channels

We have already shown (see Eq.[5]) that  $\mathsf{T}_{3\text{-LEA}} \subset \mathsf{T}_{2\text{-LEA}}$ . In this section we will investigate the inverse relation, namely whether 2-LEA depolarizing channels  $\mathcal{E}_\lambda$  are necessarily also 3-LEA channels. Under the action of  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  a general three-partite state  $\omega_{123}$  is transformed into the state

$$\begin{aligned} \omega'_{123} = & \lambda^3 \omega_{123} + \frac{1}{d^3} (1 - \lambda)^3 I_1 \otimes I_2 \otimes I_3 \\ & + \frac{1}{d} \lambda^2 (1 - \lambda) (\omega_{12} \otimes I_3 + \omega_{13} \otimes I_2 + \omega_{23} \otimes I_1) \\ & + \frac{1}{d^2} \lambda (1 - \lambda)^2 (\omega_1 \otimes I_{23} + \omega_2 \otimes I_{13} + \omega_3 \otimes I_{12}), \end{aligned}$$

where  $I_{jk} = I_j \otimes I_k$  and  $I_j$  stands for the identity operator on the  $j$ th particle. Since  $\mathcal{E}_\lambda \in \mathsf{T}_{2\text{-LEA}}$  the reduced bipartite states  $\omega'_{12}, \omega'_{13}, \omega'_{23}$  are separable. If some entanglement has left in the composite system, then it must be visible with respect to bipartite partitionings 1|23, or 2|13, or 3|12.

As before let us assume the case of qubits ( $d = 2$ ) and set  $\omega_{123} = |\text{GHZ}\rangle\langle\text{GHZ}|$ , i.e.  $\omega_{12} = \omega_{13} = \omega_{23} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \equiv \Theta$  and  $\omega_1 = \omega_2 = \omega_3 = \frac{1}{2}I$ . Thus,

$$\begin{aligned} \omega'_{\text{GHZ}} = & \lambda^3 |\text{GHZ}\rangle\langle\text{GHZ}| + \frac{(1 - \lambda)^2}{8} I \otimes I \otimes I \\ & + \frac{1}{2} \lambda^2 (1 - \lambda) (\Theta_{12} \otimes I + \Theta_{13} \otimes I + \Theta_{23} \otimes I) \end{aligned}$$

Let us consider the splitting 1|23 and define the basis elements of system 23 as follows

$$|\mathbf{0}\rangle = |00\rangle, |\mathbf{1}\rangle = |11\rangle, |\mathbf{2}\rangle = |01\rangle, |\mathbf{3}\rangle = |10\rangle. \quad (16)$$

In this basis

$$\begin{aligned} \omega'_{\text{GHZ}} = & \frac{1}{8} (1 - \lambda^2) |\mathbf{0}\rangle\langle\mathbf{0}| \otimes (I_{23} - |\mathbf{0}\rangle\langle\mathbf{0}|) \\ & + \frac{1}{8} (1 - \lambda^2) |\mathbf{1}\rangle\langle\mathbf{1}| \otimes (I_{23} - |\mathbf{1}\rangle\langle\mathbf{1}|) \\ & + \frac{1}{8} (1 + 3\lambda^2) (|\mathbf{00}\rangle\langle\mathbf{00}| + |\mathbf{11}\rangle\langle\mathbf{11}|) \\ & + \frac{1}{2} \lambda^3 (|\mathbf{00}\rangle\langle\mathbf{11}| + |\mathbf{11}\rangle\langle\mathbf{00}|). \end{aligned}$$

The last term plays the crucial role from the point of partial transposition criterion applied with respect to splitting 1|23. Let us note that due to symmetry for different splitting 2|13 and 3|12 we will derive qualitatively the same bipartite density matrix. That is, if the state  $\omega'_{\text{GHZ}}$  is entangled with respect to the splitting 1|23, then it is also entangled with respect to remaining bipartite splittings.

Among all the eigenvalues of  $\omega_{\text{GHZ}}^T$  only

$$\mu_- = \frac{1}{2} \left( \frac{1}{4} (1 - \lambda^2) - \lambda^3 \right), \quad (17)$$

is negative when  $\lambda > 0.5567$ . Important for us, is that for  $\lambda = 1/\sqrt{3}$  this eigenvalue is negative, hence the state remains entangled although the depolarizing channel is 2-LEA. Therefore, we can conclude that

$$\mathsf{T}_{3\text{-LEA}} \subsetneq \mathsf{T}_{2\text{-LEA}}. \quad (18)$$

Since partial transposition criterion is not sufficient to conclude the separability, it cannot be used to decide for which  $\lambda$  the channel  $\mathcal{E}_\lambda$  is 3-LEA and for which it is not. If  $\lambda > 0.5567$  we can safely say that  $\mathcal{E}_\lambda$  is not the 3-locally entanglement-annihilating channel. However, for smaller values we cannot exclude the possibility that the channel  $\mathcal{E}_\lambda$  does not belong to  $\mathsf{T}_{3\text{-LEA}}$  unless  $\lambda \leq 1/3$ , when the channel is entanglement-breaking.

#### 4.3. EA vs EB

In this section we shall get back to the question on relation between EA and EB channels. We have already shown that there are entanglement-breaking channels which are not entanglement-annihilating. In what follows we are interested in whether the opposite case is also possible, i.e. whether there are entanglement-annihilating channels that are entanglement-breaking. Mathematically, we are asking which of the relations  $\mathsf{T}_{\text{EA}} \subset \mathsf{T}_{\text{EB}}$ ,  $\mathsf{T}_{\text{EA}} \not\subset \mathsf{T}_{\text{EB}}$  hold. The derived results we can use to argue whether 2-LEA channels  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  considered as a channels acting on bipartite system are entanglement-breaking, or not. Certainly, they are entanglement-annihilating because  $\mathsf{T}_{2\text{-LEA}}(\mathcal{H} \otimes \mathcal{H}) \subset \mathsf{T}_{\text{EA}}(\mathcal{H} \otimes \mathcal{H})$ . We have seen that for the value  $\lambda = 1/\sqrt{3}$  the channel  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  is entanglement-annihilating. More importantly, we have also shown that for the same value  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  is not entanglement-annihilating. Let us formally write  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda \otimes \mathcal{E}_\lambda = (\mathcal{I} \otimes \mathcal{I} \otimes \mathcal{E}_\lambda)(\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda \otimes \mathcal{I})$ . Since channels of the form  $\mathcal{I} \otimes \mathcal{I} \otimes \mathcal{E}$  can only decrease the entanglement and since  $\mathcal{E}_\lambda^{\otimes 3}[|\text{GHZ}\rangle\langle\text{GHZ}|]$  is entangled for  $\lambda = 1/\sqrt{3}$ , we can conclude that also  $(\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda) \otimes \mathcal{I}[|\text{GHZ}\rangle\langle\text{GHZ}|]$  is entangled. But this is in contradiction with the assumption that  $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$  is entanglement-breaking. Based on this example we can conclude that

$$\mathsf{T}_{\text{EA}} \not\subset \mathsf{T}_{\text{EB}}, \quad (19)$$

that is, there are entanglement-annihilating channels which are not entanglement-breaking.

## 5. Summary

In this paper we introduced the concept of entanglement-annihilating channels as the channels that completely destroy any entanglement within the systems they act on. We investigated the structural properties of the set of these channels and its relation to

the set of entanglement-breaking channels  $\mathsf{T}_{\text{EB}}$ , i.e. channels that completely destroy entanglement between the subsystem they act on and the rest of the composite system (see Fig. 2). In particular, we have shown that

$$\mathsf{T}_{\text{EA}} \cap \mathsf{T}_{\text{EB}} \neq \emptyset, \quad (20)$$

$$\mathsf{T}_{\text{EB}} \not\subset \mathsf{T}_{\text{EA}} \not\subset \mathsf{T}_{\text{EB}}. \quad (21)$$

That is, there are channels which are simultaneously entanglement-breaking and entanglement-annihilating, but also channels possessing only one of this features. The set of entanglement-annihilating channels  $\mathsf{T}_{\text{EA}}$  is convex. Moreover, a composition of an entanglement-annihilating channel and an arbitrary channel results in an entanglement-annihilating channel, i.e. the property of being entanglement-annihilating is preserved under channel composition.

One of the above relations we were able to prove by analyzing the family of local depolarizing channels. We defined the so-called  $k$ -local channels as channels of the form  $\mathcal{E} \otimes \cdots \otimes \mathcal{E}$ . That is, the same noise  $\mathcal{E}$  is applied on each individual subsystem forming a composite  $k$ -partite system. We investigated when a single-particle channel  $\mathcal{E}$  constitutes a  $k$ -locally entanglement-annihilating channel ( $k$ -LEA), or a  $k$ -locally entanglement-breaking channel ( $k$ -LEB). In particular, for depolarizing qubit channels we found that for  $\lambda \leq 1/\sqrt{3}$  the channel is 2-locally entanglement annihilating, while for  $\lambda > 1/3$  it is not locally entanglement-breaking for any  $k$ . Moreover, for  $\lambda > 0.5567$  the qubit depolarizing channel is not  $k$ -LEA for all  $k \geq 3$ . We found the following set relations

$$\mathsf{T}_{\text{EB}}^1 = \mathsf{T}_{2\text{-LEB}} = \mathsf{T}_{3\text{-LEB}} = \cdots = \mathsf{T}_{\infty\text{-LEB}}, \quad (22)$$

$$\mathsf{T}_{\text{EB}}^1 \subset \mathsf{T}_{\infty\text{-LEA}} \subset \cdots \subset \mathsf{T}_{3\text{-LEA}} \subsetneq \mathsf{T}_{2\text{-LEA}}, \quad (23)$$

where  $\mathsf{T}_{\text{EB}}^1$  is the set of entanglement-breaking channels of a single particle.

The introduced concept of entanglement-annihilating channels opens several interesting mathematical and physical questions related to generic properties of entanglement dynamics. For example, we left open the problem of complete characterization of entanglement-annihilating channels. For practical purposes, it would be also be of interest to find an efficient testing algorithm for entanglement-annihilating channels.

## Acknowledgments

We acknowledge financial support of the European Union project HIP FP7-ICT-2007-C-221889, and of the projects VEGA-2/0092/09, CE SAS QUTE and MSM0021622419. We also thank Daniel Nagaj for his comments on the manuscript.

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